

# Demazure modules and vertex models: the $\widehat{sl}(2)$ case

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## Abstract

We characterize, in the case of  $\widehat{sl}(2)$ , the crystal base of the Demazure module  $E_w(\Lambda)$  in terms of extended Young diagrams or paths for any dominant integral weight  $\Lambda$  and Weyl group element  $w$ . Its character is evaluated via two expressions, 'bosonic' and 'fermionic'.

## 0 Introduction

The purpose of this work is to characterize the crystal bases of the Demazure modules of  $\widehat{sl}(2)$  in terms of 'paths', or extended Young diagrams, and obtain explicit expressions for their full characters. There are two ways to motivate this

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work: one can motivate it from the viewpoint of certain recent developments in mathematical physics, or from the viewpoint of representation theory.

## 0.1 1-point functions in exactly solvable models

Certain physical quantities in exactly solvable two-dimensional lattice models, namely the so-called '1-point functions' can be evaluated using Baxter's corner transfer matrix method.<sup>1</sup> This method reduces the computation of 1-point functions to a computation of a weighted sum. This sum is over combinatorial objects called 'paths': 1-dimensional configurations, defined on the half-line. They will be discussed in detail in the sequel. The weight of a path is the evaluation, on that path, of an 'energy' functional defined on the set of all paths. The weighted sum is the generating function of the number of paths of a certain weight.

One way to evaluate this generating function is by solving a recursion relation for the generating function that counts the number of paths defined on a finite segment, of length  $L$ , of the half-line. The recursion is with respect to  $L$ . The  $L \rightarrow \infty$  limit of the result, which is the generating function of all paths, turns out to be the character of a highest weight module of an infinite-dimensional algebra: an affine, or Virasoro algebra.<sup>2</sup> The result obtained by solving a recursion relation typically has a form that, for physical reasons, is called 'bosonic'.<sup>3</sup>

## 0.2 Rogers-Ramanujan-type $q$ -series identities

More recently, the same physical objects were evaluated in a completely different way, using Bethe Ansatz methods [3]. In that case, the results turn out to have a completely different form, that, for the same physical reasons as above, is referred to as 'fermionic'.

Equating the bosonic and fermionic expressions for the same objects, one obtains Rogers-Ramanujan-type identities between  $q$ -series. Strictly speaking, such identities are conjectures, since the methods used to obtain them are indirect, and involve certain 'physical assumptions'.<sup>4</sup> What one needs are direct proofs.

## 0.3 Schur-type polynomial identities

One method to obtain direct proofs, which dates back to Schur, is to work at the level of the generating functions of all paths that live on a segment of the half-line of length  $L$ . These are polynomials that depend on  $L$ . Once we can

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<sup>1</sup>For an introduction to exactly solvable models, and to the corner transfer matrix method, we refer to [1].

<sup>2</sup>For an introduction to the algebraic approach to exactly solvable models, we refer to [2].

<sup>3</sup>For an explanation of the origin of this terminology, we refer to [3].

<sup>4</sup>What we have in mind are the various assumptions involved in the derivation of the corner transfer matrix method, and the 'string hypothesis'.

prove the boson-fermion identities for finite  $L$ , we take the limit  $L \rightarrow \infty$ , and obtain the original  $q$ -series identities.

The following question now arises: If the original infinite  $q$ -series are characters of highest weight modules of infinite-dimensional algebras, are the finite- $L$  polynomials the characters of anything? Do they have a meaning in mathematics? Or are they just convenient objects that appear in intermediate steps?

The reason why this question arises is that rigorous proofs in exactly solvable models are typically obstructed by the fact that we have to deal with infinite series, and infinite-dimensional quantities. The reason why we typically have to do that, is that only in the limit  $L \rightarrow \infty$  do these model exhibit invariance under infinite-dimensional algebras, and this invariance is an essential ingredient in solvability. Thus we have the following problem: we need to work in the  $L \rightarrow \infty$  limit in order to be able to obtain exact solutions, but in that limit we cannot provide rigorous proofs.

This is the reason why it is very interesting to try to understand as much of the structure of exactly solvable models as possible at the level of finite  $L$ . The hope is that we can find algebraic structures at finite  $L$  that are sufficiently strong to provide rigorous proofs and solutions. A typical object to investigate are the  $L$ -restricted generating functions.

What we find in this work, is that these objects do have a mathematical significance, or are directly related to objects that do. We find that the  $L$ -restricted generating functions are closely related to the characters of Demazure modules.

## 0.4 Demazure modules

From the viewpoint of representation theory, we can motivate our work in the evaluation of the characters of Demazure modules. The Demazure module is characterized by its highest weight  $\Lambda$  and Weyl group element  $w$ , and is denoted by  $E_w(\Lambda)$ . In [4, 5], a formula for computing the Demazure character  $chE_w(\Lambda) = \sum_{\mu \in P} \dim(E_w(\Lambda))_{\mu} e^{\mu}$  is given as follows. For  $\mu \in P, i = 0, 1$ , define the operator  $D_i : \mathbb{Z}[P] \rightarrow \mathbb{Z}[P]$  by

$$D_i(e^{\mu}) = \frac{e^{\mu+\rho} - e^{r_i(\mu+\rho)}}{1 - e^{-\alpha_i}} e^{-\rho}.$$

Let  $w = r_{i_n} \cdots r_{i_2} r_{i_1} \in W$  be a reduced expression. For the terminologies of  $\widehat{sl}(2)$  see subsection 1.1. Then, the Demazure character formula states that

$$chE_w(\Lambda) = D_{i_n} \cdots D_{i_2} D_{i_1}(e^{\Lambda}).$$

As elegant as this character formula is, it is anything but combinatorial. Hence using this formula it is very difficult to get any information about specific weight spaces. In [6], Sanderson used Littelmann's path model [7] and gave nice expressions for the 'real' characters of  $E_w(\Lambda)$  for all  $\Lambda$  and the 'principal' characters

of  $E_w(\Lambda)$  for  $\Lambda = s\Lambda_0$ . We point out that the real (resp. principal) character of  $E_w(\Lambda)$  is the specialization of  $e^{-\Lambda}chE_w(\Lambda)$  where  $e^{-\alpha_0} = q, e^{-\delta} = 1$  (resp.  $e^{-\alpha_0} = q, e^{-\delta} = q^2$ ).

## 0.5 Crystal bases

In 1990 Kashiwara brought a notion of crystal base in the representation theory of the quantum group [8]. This notion is quite powerful in the combinatorial aspect of representation theory. As an example, we find [9] in which the Littlewood-Richardson type rules for the classical Lie algebras are given in a purely combinatorial way. Kashiwara also obtained the crystal bases of the Demazure modules as subsets of those of the corresponding highest weight modules [10]. His algorithm is given in a recursive way. Consequently he got a new proof of the Demazure character formula.

Since we know that in the case of affine Lie algebra the crystal base of the highest weight module is described in terms of paths [11, 12], it is natural to try to characterize the crystal base of the Demazure module in this language. For the affine algebra of type A we have yet another notion to describe the crystal base, that is, extended Young diagram [13, 14].

## 0.6 Main results

What we have done in our work is the characterization of Demazure crystal bases in terms of extended Young diagrams (Theorem 2). One of the advantages to do so is that we can make the most of earlier efforts toward the evaluations of so called 1-dimensional configuration sums [15, 16]. Recent results on fermionic expressions are also available. We would like to note here that after we had obtained the fermionic expression (Theorem 3), we came to know Schilling also had the same result [17]. However, we have included our proof since it is different from that of Schilling. As a corollary of the characterization theorem, we evaluate the characters of Demazure modules (Theorem 4 and 5).

## 0.7 Outline of paper

In section 1, we define the basic combinatorial objects in this work: paths and extended Young diagrams. We then describe the crystal structure of the integrable highest weight module of  $U_q(\widehat{sl}(2))$ . In section 2, we introduce the Demazure modules, and characterize their crystals via paths or extended Young diagrams. In section 3, we review 1-dimensional configuration sums and rewrite them into fermionic forms. Using them we can express Demazure characters. We also discuss some specializations and compare them with Sanderson's results. Section 4 contains a short discussion.

# 1 Paths and extended Young diagrams

In this section we recall the basic combinatorial objects used in this work: the paths, and extended Young diagrams. Next, we outline the realization of the crystal base of an integrable  $\widehat{sl}(2)$ -module in terms of these objects. We also list some properties of the crystal base which we will need in the sequel.

## 1.1 Preliminaries

Consider the affine Lie algebra  $\widehat{sl}(2)$  [18]. Let  $\{\alpha_0, \alpha_1\}$ ,  $\{h_0, h_1\}$  and  $\{\Lambda_0, \Lambda_1\}$  be the simple roots, simple co-roots and fundamental weights respectively. They satisfy  $\Lambda_i(h_j) = \delta_{ij}$  and  $\alpha_i(h_j) = 2(-1)^{1+\delta_{ij}}$  for  $i, j = 0, 1$ .  $\delta = \alpha_0 + \alpha_1$ ,  $c = h_0 + h_1$  are the null root and canonical central element, respectively. The sets  $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\delta$ , and  $Q = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1$  are the weight and root lattices, respectively. Let  $\rho \in \mathfrak{h}^*$  be such that  $\rho(h_i) = 1$  for  $i = 0, 1$ .

Let  $U_q(\widehat{sl}(2))$  denote the quantized universal enveloping algebra associated with  $\widehat{sl}(2)$ . For its precise definition and Hopf algebra structure, we refer to [14]. For  $\Lambda \in P$ ,  $\Lambda(c)$  is the level of  $\Lambda$ . The set  $P^+ = \{\Lambda \in P \mid \Lambda(h_i) \geq 0, i = 0, 1\}$  is the set of dominant weights. For  $\Lambda \in P^+$ , let  $V(\Lambda)$  denote the unique (up to isomorphism) integrable highest weight  $U_q(\widehat{sl}(2))$ -module. Since  $V(\Lambda + k\delta) \simeq V(\Lambda) \otimes V(k\delta)$  and  $\dim V(k\delta) = 1$ , it suffices to assume that  $\Lambda = s\Lambda_0 + t\Lambda_1$  for some  $s, t \in \mathbb{Z}_{\geq 0}$ . Let  $\Lambda(c) = s + t = k$  be the level of  $\Lambda$ .

## 1.2 Paths

For convenience, we extend the subscript  $i$  of  $\Lambda_i$  to  $i \in \mathbb{Z}$ , by setting  $\Lambda_i = \Lambda_{i'}$  for  $i \equiv i' \pmod{2}$ . We also set  $\widehat{i} = \Lambda_{i+1} - \Lambda_i$  ( $i = 0, 1$ ). For  $k \in \mathbb{Z}_{\geq 0}$ , we define a set of level  $k$  weights  $P_k$  by  $P_k = \{a_0\Lambda_0 + a_1\Lambda_1 \mid a_0, a_1 \in \mathbb{Z}, a_0 + a_1 = k\}$ .

**Definition 1 (path on  $P_k$ )** Fixing  $k, L \in \mathbb{Z}_{\geq 0}$  we define a path  $p$  of length  $L$  as a sequence  $p = (p_0, \dots, p_L, p_{L+1})$ , with all  $p_i \in P_k$  and  $p_{i+1} - p_i \in \{k\widehat{0}, (k-1)\widehat{0} + \widehat{1}, \dots, k\widehat{1}\}$ .

**Definition 2 (set of paths  $\mathcal{P}_L(\Lambda)$ )** For  $L \in \mathbb{Z}_{\geq 0}$  and a dominant weight  $\Lambda = s\Lambda_0 + t\Lambda_1$  ( $s + t = k$ ), we set

$$\mathcal{P}_L(\Lambda) = \left\{ p = (p_0, \dots, p_L, p_{L+1}) \mid \begin{array}{l} p \text{ is a path on } P_k, \\ p_i = s\Lambda_i + t\Lambda_{i+1} \text{ for } i = L, L+1 \end{array} \right\}.$$

To the set  $\mathcal{P}_L(\Lambda)$ , we associate a special path  $\bar{p}$ , called the ground-state path, defined as follows:

**Definition 3 (ground-state path  $\bar{p}$ )**

$$\bar{p} = (\bar{p}_0, \dots, \bar{p}_L, \bar{p}_{L+1}), \quad \bar{p}_i = s\Lambda_i + t\Lambda_{i+1} \text{ for all } i.$$

The relevance of the ground-state path will become clear in the sequel.

For  $k \in \mathbb{Z}_{\geq 0}$  set  $\mathcal{A}_k^+ = \{a_0\epsilon_0 + a_1\epsilon_1 \mid a_0, a_1 \in \mathbb{Z}_{\geq 0}, a_0 + a_1 = k\}$ . Here we consider  $\epsilon_0$  and  $\epsilon_1$  as symbols. Just as in the case of  $\Lambda_i$ , we extend the subscript  $i$  of  $\epsilon_i$  to  $i \in \mathbb{Z}$  by setting  $\epsilon_i = \epsilon_{i'}$  for  $i \equiv i' \pmod{2}$ . We encode a path in terms of a sequence of elements in  $\mathcal{A}_k^+$  as follows:

**Definition 4 (sequence of elements  $\iota(p)$ )** Let  $\Lambda = s\Lambda_0 + t\Lambda_1$  ( $s+t = k$ ). For a path  $p = (p_0, \dots, p_L, p_{L+1}) \in \mathcal{P}_L(\Lambda)$  we define a sequence  $\iota(p) = (\iota(p)_0, \dots, \iota(p)_L)$  by  $\iota(p)_i = m_i\epsilon_0 + (k - m_i)\epsilon_1 \in \mathcal{A}_k^+$ , where for each  $i$ ,  $m_i$  is determined by  $p_{i+1} - p_i = m_i\hat{0} + (k - m_i)\hat{1}$ .

For the ground-state path  $\bar{p}$ ,  $\iota(\bar{p})$  is given by  $\iota(\bar{p})_i = s\epsilon_i + t\epsilon_{i+1}$ . We remark that the data  $\iota(p)$  uniquely determine  $p \in \mathcal{P}_L(\Lambda)$ .

Define a function  $\mu : \mathcal{A}_k^+ \longrightarrow \{0, 1, \dots, k\}$  by

$$\mu(m\epsilon_0 + (k - m)\epsilon_1) = m.$$

Using  $\mu$ , we define the energy, and the weight of a path  $p$  as follows:

**Definition 5 (energy of a path  $E(p)$ )**

$$E(p) = \sum_{j=1}^L j \left( H(\iota(p)_{j-1}, \iota(p)_j) - H(\iota(\bar{p})_{j-1}, \iota(\bar{p})_j) \right),$$

where

$$H(\epsilon, \epsilon') = \max(k - \mu(\epsilon), \mu(\epsilon')) \quad \text{for } \epsilon, \epsilon' \in \mathcal{A}_k^+. \quad (1)$$

**Definition 6 (weight of a path  $wt\ p$ )**

$$wt\ p = p_0 - E(p)\delta.$$

We note that for the ground-state path  $\bar{p} \in \mathcal{P}_L(\Lambda)$  we have  $E(\bar{p}) = 0$  and  $wt\ \bar{p} = \Lambda$ . We will need the following lemma in the sequel.

**Lemma 1** Let  $\Lambda = s\Lambda_0 + t\Lambda_1$  ( $s+t = k$ ). For the ground-state path  $\bar{p} \in \mathcal{P}_L(\Lambda)$ , we have

$$\sum_{j=1}^L j H(\iota(\bar{p})_{j-1}, \iota(\bar{p})_j) = \left( \frac{L + \epsilon^{(L)}}{2} \right)^2 k + (-1)^{\epsilon^{(L)}} \frac{L + \epsilon^{(L)}}{2} s.$$

Here  $\epsilon^{(L)}$  is defined by

$$\epsilon^{(L)} = \begin{cases} 0 & (L : \text{even}), \\ 1 & (L : \text{odd}). \end{cases} \quad (2)$$

*Proof.* Note that  $\iota(\bar{p}_j) = s\epsilon_j + t\epsilon_{j+1}$ . From (1) we obtain the following expression to calculate:

$$\sum_{\substack{j=1 \\ j:\text{odd}}}^L jt + \sum_{\substack{j=1 \\ j:\text{even}}}^L js,$$

which is evaluated easily. ■

So far, we fixed the length of paths  $L$ . Next, we consider two lengths  $L$  and  $L'$  ( $L \leq L'$ ). Let  $\Lambda = s\Lambda_0 + t\Lambda_1$ . Then there is an injection given by

$$\begin{aligned} \mathcal{P}_L(\Lambda) &\longrightarrow \mathcal{P}_{L'}(\Lambda) \\ p = (p_0, \dots, p_L, p_{L+1}) &\mapsto p' = (p'_0, \dots, p'_{L'}, p'_{L'+1}), \end{aligned}$$

where  $p'_i = p_i$  ( $0 \leq i \leq L+1$ ),  $= s\Lambda_i + t\Lambda_{i+1}$  ( $L+1 \leq i \leq L'+1$ ). Let  $\bar{p}'$  be the ground-state path of  $\mathcal{P}_{L'}(\Lambda)$ . The image of  $\mathcal{P}_L(\Lambda)$  in  $\mathcal{P}_{L'}(\Lambda)$  is characterized by the paths  $\{p'\}$  such that  $p'_i = \bar{p}'$  ( $i \geq L$ ). Note that the energy and weight of a path are unchanged by this injection. We have the following inductive system:

$$\mathcal{P}_0(\Lambda) \longrightarrow \mathcal{P}_1(\Lambda) \longrightarrow \dots \longrightarrow \mathcal{P}_L(\Lambda) \longrightarrow \dots$$

We define the set of all paths  $\mathcal{P}(\Lambda)$  as follows:

**Definition 7** ( $\mathcal{P}(\Lambda)$ )

$$\mathcal{P}(\Lambda) = \lim_{\rightarrow} \mathcal{P}_L(\Lambda).$$

### 1.3 Extended Young diagrams

An extended Young diagram  $Y$  is an infinite sequence  $(y_j)_{j \geq 0}$  of integers, such that  $y_j \leq y_{j+1}$  for all  $j$  and  $y_j = y_\infty$  (a fixed integer) for sufficiently large  $j$  [15, 13, 14]. In other words, the sequence stabilizes after a finite (though arbitrarily large) number of elements. The number  $y_\infty$  is called the ‘charge’ of  $Y$ . In this paper, we only consider extended Young diagrams of charge  $\gamma = 0$  or 1.

We can view an extended Young diagram  $Y = (y_j)_{j \geq 0}$  as a diagram drawn on the lattice in the right-half plane with sites  $\{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid m \geq 0\}$  where  $y_j$  denotes the depth of the  $j$ -th column. If  $y_j \neq y_{j+1}$ , for some  $j$ , then there will be concave (‘ $\lceil$ ’), and convex (‘ $\rfloor$ ’) corners. A corner located at site  $(m, n)$  has diagonal number  $d = m + n$ .

#### 1.3.1 A coloring scheme

We assign each corners of  $Y = (y_j)_{j \geq 0}$  one of two colors: 0 = ‘white’, or 1 = ‘black’ as follows: A  $d$ -diagonal corner is white (resp. black), if  $d$  is even (resp. odd). A white (resp. black) corner will also be called a 0-corner (resp. 1-corner).

Thus, we can also view an extended Young diagram  $Y = (y_j)_{j \geq 0}$  of charge 0 (resp. 1) as a usual Young diagram, but with its nodes alternately colored ‘white’ and ‘black’, such that the top left-most node has color ‘white’ (resp. ‘black’).

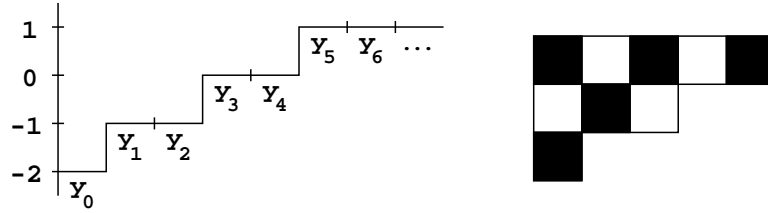
We define the weight  $\text{wt } Y$  of an extended Young diagram  $Y$  of charge  $i$  ( $= 0$  or 1) to be

$$\text{wt } Y = \Lambda_i - k_0 \alpha_0 - k_1 \alpha_1,$$

where  $k_0$  is the number of white nodes in  $Y$ , and  $k_1$  is the number of black nodes. We also define the ‘width of  $Y$ ’,  $|Y|$ , to be the number of nodes in the first row of  $Y$ . Equivalently,  $|Y| = L$  if and only if  $y_L = y_\infty (= \text{charge of } Y)$  but  $y_{L-1} < y_L$ .

### 1.3.2 Example

Consider the extended Young diagram  $Y = (y_j)_{j \geq 0} = (-2, -1, -1, 0, 0, 1, 1, 1, \dots)$  of charge 1.  $Y$  has convex 1-corners at sites  $(1, -2)$  and  $(5, 0)$ , convex 0-corner at site  $(3, -1)$ , concave 1-corner at site  $(3, 0)$  and a concave 0-corner at site  $(1, -1)$ . Also note that  $\text{wt } Y = \Lambda_1 - 4\alpha_0 - 5\alpha_1$  and  $|Y| = 5$ .



## 1.4 From paths to extended Young diagrams

Let us define a ‘pattern’ to be a map

$$\begin{aligned} t : \mathbb{Z} \times \mathbb{Z}_{\geq 0} &\longrightarrow \mathbb{Z} \\ (i, j) &\longmapsto t_{ij} \end{aligned}$$

such that

- (i) for all  $i$ ,  $(t_{ij})_{j \geq 0}$  is an extended Young diagram,
- (ii)  $t_{ij} \leq t_{i+1, j}$  for all  $i$  and  $j$ ,
- (iii)  $t_{i+k, j} = t_{ij} + 2$  for all  $i$  and  $j$ .

We say the pattern  $t$  is normalized if  $0 \leq \gamma_1 \leq \dots \leq \gamma_k < 2$ , where  $\gamma_i = t_{i\infty}$  is the charge of  $(t_{ij})_{j \geq 0}$ . We call  $\gamma = (\gamma_1, \dots, \gamma_k)$  the charge of  $t$ . Viewing the condition (iii) we can identify the pattern  $t$  with a  $k$ -tuple  $\mathbf{Y} = (Y_1, \dots, Y_k)$  of extended Young diagrams  $Y_i = (t_{ij})_{j \geq 0}$ . In terms of extended Young diagrams, (ii) is equivalent to the following inclusion rule:

$$Y_1 \supset Y_2 \supset \dots \supset Y_k \supset Y_1[2] \quad (3)$$

where  $Y_1[2]$  denotes the extended Young diagram which is obtained by shifting  $Y_1$  upward two units on the lattice in the right-half plane. Let  $\Lambda = s\Lambda_0 + t\Lambda_1$  ( $s + t = k$ ). Denote the set of patterns with charge  $\gamma = (\underbrace{0, \dots, 0}_s, \underbrace{1, \dots, 1}_t)$  by

$\mathcal{T}(\Lambda)$ . Then we have a map

$$\begin{aligned} \pi : \mathcal{T}(\Lambda) &\longrightarrow \mathcal{P}(\Lambda) \\ t = (t_{ij}) &\longmapsto p, \end{aligned}$$

where  $p$  is determined from  $\iota(p)_j = \epsilon_{t_{1j}+j} + \dots + \epsilon_{t_{kj}+j}$ . For a path  $p \in \mathcal{P}(\Lambda)$  we say  $t$  is a ‘lift’ of  $p$  if  $t \in \pi^{-1}(p)$ . Among lifts of  $p \in \mathcal{P}(\Lambda)$  there exists a unique normalized lift  $t = (t_{ij}) \in \mathcal{T}(\Lambda)$  such that  $t_{ij} \geq t'_{ij}$  for all  $i, j$  for any  $t' = (t'_{ij}) \in \pi^{-1}(p)$  (Proposition 3.4 in [14]).  $t(p)$  is called the ‘highest lift’ of  $p$ . Let  $\mathbf{Y} = (Y_1, \dots, Y_k)$  be the  $k$ -tuple of extended Young diagrams corresponding to the highest lift of  $p$ . Then we have (Theorem 5.7 in [15])

$$wt\ p = wt\ Y_1 + \dots + wt\ Y_k.$$

## 1.5 The crystal $B(\Lambda)$

Let  $(L(\lambda), B(\lambda))$  be the crystal base of  $V(\lambda)$ . In this subsection we give the combinatorial rules used to construct the crystal  $B(\Lambda)$ , following the work of [14, 13].

### 1.5.1 The rules

Let  $\Lambda = s\Lambda_0 + t\Lambda_1 \in P^+$  of level  $k = s + t$ . We order the elements of  $\mathbb{Z} \times \{1, 2, \dots, k\}$  as follows. For  $(d, j), (d', j') \in \mathbb{Z} \times \{1, 2, \dots, k\}$  we say

$$(d, j) > (d', j') \text{ if and only if } d > d' \text{ or } d = d' \text{ and } j < j'.$$

Let  $\mathcal{Y}(\Lambda)$  denote the set of  $k$ -tuples of extended Young diagrams  $\mathbf{Y} = (Y_1, \dots, Y_s, Y_{s+1}, \dots, Y_k)$  such that  $Y_j$  has charge 0 (resp. 1) for  $1 \leq j \leq s$  (resp.  $s+1 \leq j \leq k$ ). For  $\mathbf{Y} \in \mathcal{Y}(\Lambda)$  and fixed  $i$  ( $= 0$  or  $1$ ), we define the  $i$ -signature of  $\mathbf{Y}$  to be the sequence  $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)^5$  such that

<sup>5</sup>Note that we distinguish  $\varepsilon_j$  from  $\epsilon_j$  which has appeared in Definition 5.

- A1  $\sum_{j=1}^k \# \{i\text{-corners of } Y_j\} = m$ ,
- A2 each  $\varepsilon_r$  is either 0 or 1,
- A3 we can define  $j(r)$  ( $1 \leq j(r) \leq k$ ) and  $d(r)$  in such a way that  $Y_{j(r)}$  has a  $d(r)$ -diagonal  $i$ -corner,
- A4 if  $\varepsilon_r = 0$  (resp. 1) then  $Y_{j(r)}$  has a  $d(r)$ -diagonal concave (resp. convex)  $i$ -corner,
- A5 if  $r_1 < r_2$ , then  $(d(r_1), j(r_1)) > (d(r_2), j(r_2))$ .

For fixed  $i$ -signature  $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$  we partition the set  $\{1, 2, \dots, m\} = J \sqcup K_1 \sqcup \dots \sqcup K_r$  into disjoint subsets by the following procedure:

- B1 if there is no  $j$  such that  $(\varepsilon_j, \varepsilon_{j+1}) = (0, 1)$  define  $J = \{1, 2, \dots, m\}$ ,
- B2 if there is some  $j$  such that  $(\varepsilon_j, \varepsilon_{j+1}) = (0, 1)$  define  $K_1 = \{j, j+1\}$ ,
- B3 apply B1 and B2 above to  $\{1, 2, \dots, m\} \setminus K_1$  to choose  $J$  or  $K_2$  and repeat this as necessary to choose  $J$  and  $K_1, \dots, K_r$ .

Let  $\varepsilon_J = (\varepsilon_{j_1}, \dots, \varepsilon_{j_r})$ , where  $J = \{j_1, \dots, j_r\}$  and  $j_1 < \dots < j_r$ . We call a 0 or 1 in the  $i$ -signature ‘relevant’ if and only if it is in  $\varepsilon_J$ .

For fixed  $i \in \{0, 1\}$  and  $\mathbf{Y}, \mathbf{Y}' \in \mathcal{Y}(\Lambda)$  suppose that the following conditions hold:

- C1  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$  and  $\varepsilon' = (\varepsilon'_1, \dots, \varepsilon'_m)$  are the  $i$ -signatures of  $\mathbf{Y}$  and  $\mathbf{Y}'$  respectively,
- C2 the partition  $\{1, 2, \dots, m\} = J \sqcup K_1 \sqcup \dots \sqcup K_r$  is same for both  $\varepsilon$  and  $\varepsilon'$ ,
- C3 there exists  $l \in J$  such that  $\varepsilon_l = 0, \varepsilon'_l = 1$ , and  $\varepsilon_j = \varepsilon'_j = 1$  (resp. 0) if  $j \in J$  and  $j < l$  (resp.  $j > l$ ).

We define  $\tilde{f}_i \mathbf{Y} = \mathbf{Y}'$  and  $\tilde{e}_i \mathbf{Y}' = \mathbf{Y}$  if and only if the above conditions hold. If there exists no such  $\mathbf{Y}'$  (resp.  $\mathbf{Y}$ ) for  $\mathbf{Y}$  (resp.  $\mathbf{Y}'$ ), we define  $\tilde{f}_i \mathbf{Y} = 0$  (resp.  $\tilde{e}_i \mathbf{Y}' = 0$ ).

### 1.5.2 Example

Let  $\Lambda = \Lambda_0 + \Lambda_1$  and take  $\mathbf{Y} = (Y_1, Y_2) \in \mathcal{Y}(\Lambda)$ , where  $Y_1 = (-2, -2, -1, -1, 0, 0, \dots)$  is of charge 0 and  $Y_2 = (-1, 0, 0, 1, 1, \dots)$  is of charge 1.

$$\mathbf{Y} = (Y_1, Y_2) = \left( \begin{array}{cc} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{array}, \begin{array}{cc} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{array} \right).$$

The 0-signature of  $\mathbf{Y}$  is  $\varepsilon = (0, 0, 1, 1, 0)$ . So using rule B  $\varepsilon_J = (0)$ . Hence by using rule C we have

$$\tilde{f}_0 \mathbf{Y} = \mathbf{Y}' = \left( \begin{array}{cc} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{array}, \begin{array}{cc} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{array} \right) \quad \text{and} \quad \tilde{e}_0 \mathbf{Y} = 0.$$

However, the 1-signature of  $\mathbf{Y}$  is  $\varepsilon = (1, 1, 0, 0, 0)$  and by rule B  $\varepsilon_J = \varepsilon$ . So by rule C we have

$$\tilde{f}_1 \mathbf{Y} = \left( \begin{array}{cc} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{array}, \begin{array}{cc} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{array} \right) \quad \text{and} \quad \tilde{e}_1 \mathbf{Y} = \left( \begin{array}{cc} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{array}, \begin{array}{cc} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{array} \right).$$

Thus, in general  $\tilde{e}_i, \tilde{f}_i (i = 0, 1)$  are well-defined maps from  $\mathcal{Y}(\Lambda)$  to  $\mathcal{Y}(\Lambda) \sqcup \{0\}$ . Now let  $\Phi = (\phi_1, \dots, \phi_s, \phi_{s+1}, \dots, \phi_k) \in \mathcal{Y}(\Lambda)$  be the  $k$ -tuple of empty extended Young diagrams where  $\phi_j$  has charge 0 (resp. 1) for  $1 \leq j \leq s$  (resp.  $s+1 \leq j \leq k$ ). Note that  $wt \Phi = \Lambda = s\Lambda_0 + t\Lambda_1$ , and  $\tilde{e}_i \Phi = 0$  for  $i = 0, 1$ . We call  $\Phi$  the highest weight vector (or vacuum) of weight  $\Lambda$ .

We now define the set  $B(\Lambda) \subset \mathcal{Y}(\Lambda)$  as:

$$B(\Lambda) = \{\tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_k} \Phi \mid k \geq 0, i_j = 0, 1\} \setminus \{0\}.$$

For  $\mathbf{Y} \in B(\Lambda), i = 0, 1$ , define

$$\begin{aligned} \varepsilon_i(\mathbf{Y}) &= \max\{m \geq 0 \mid \tilde{e}_i^m \mathbf{Y} \in B(\Lambda)\}, \\ \varphi_i(\mathbf{Y}) &= \max\{m \geq 0 \mid \tilde{f}_i^m \mathbf{Y} \in B(\Lambda)\}. \end{aligned}$$

Thus  $\varepsilon_i^6, \varphi_i : B(\Lambda) \longrightarrow \mathbb{Z}$ . We also have the map  $wt : B(\Lambda) \longrightarrow P$  given by  $\mathbf{Y} \mapsto wt \mathbf{Y}$ . The set  $B(\Lambda)$  equipped with these maps  $\tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i$  and  $wt$  turns out to be the ‘crystal’ (see [10, 20]) associated with the integrable highest weight module  $V(\Lambda)$ . In particular, for  $i = 0, 1$  we have

$$\tilde{e}_i(B(\Lambda)) \subseteq B(\Lambda) \sqcup \{0\} \quad \text{and} \quad \tilde{f}_i(B(\Lambda)) \subseteq B(\Lambda) \sqcup \{0\}.$$

Moreover,  $B(\Lambda)$  turns out to be the set of  $k$ -tuples of extended Young diagrams corresponding to the highest lifts of  $\mathcal{P}(\Lambda)$  (Proposition 3.12 in [14]).

The crystal  $B(\Lambda)$  has the structure of an oriented colored graph with elements of  $B(\Lambda)$  as the set of vertices and for  $\mathbf{Y}, \mathbf{Y}' \in B(\Lambda), \mathbf{Y} \xrightarrow{i} \mathbf{Y}' (i = 0, 1)$

---

<sup>6</sup>We use this new definition of  $\varepsilon_i$  only here. We hope this will not cause any confusion.

if and only if  $\tilde{f}_i \mathbf{Y} = \mathbf{Y}'$ . As a graph  $B(\Lambda)$  is connected. In particular, for  $\mathbf{Y} \in B(\Lambda)$  we have (see Proposition 3.11 in [14]),

$$\mathbf{Y} \neq \Phi \implies \tilde{e}_0 \mathbf{Y} \neq 0 \quad \text{or} \quad \tilde{e}_1 \mathbf{Y} \neq 0. \quad (4)$$

As a consequence of (3), we have

$$|Y_1| \geq \dots \geq |Y_s| \quad \text{and} \quad |Y_{s+1}| \geq \dots \geq |Y_k|.$$

Now we have the following simple but important observations.

**Proposition 1** *Let  $\mathbf{Y} = (Y_1, \dots, Y_s, Y_{s+1}, \dots, Y_k) \in B(\Lambda)$  and  $\mathbf{Y} \neq \Phi$ . Then  $|Y_1| \neq |Y_{s+1}|$ .*

*Proof.* Suppose  $|Y_1| = |Y_{s+1}|$ . Since  $Y_1$  has charge 0 and  $Y_{s+1}$  has charge 1, the last nodes in the first rows of  $Y_1$  and  $Y_{s+1}$  will have opposite color. Also by (3), the second rows of  $Y_1$  and  $Y_{s+1}$  have at least one node less than their first rows. Consequently, we have two convex corners of opposite colors corresponding to the last nodes in the first rows of  $Y_1$  and  $Y_{s+1}$ . One of these convex corners will contribute a 1 to the 0-signature of  $\mathbf{Y}$  and the other will contribute a 1 to the 1-signature of  $\mathbf{Y}$ . However, in both cases this 1 will not be relevant as it will be preceded by at least one 0 in each signature. Therefore, by applying  $\tilde{e}_0$  and  $\tilde{e}_1$  successively we will obtain  $\mathbf{Y}' = (Y'_1, \dots, Y'_s, Y'_{s+1}, \dots, Y'_k) \in B(\Lambda)$  such that

$$\begin{aligned} \mathbf{Y}' &= \tilde{e}_{i_m} \tilde{e}_{i_{m-1}} \dots \tilde{e}_{i_1} \mathbf{Y} \text{ for some } m \text{ and } i_j \in \{0, 1\}, \\ \tilde{e}_0 \mathbf{Y}' &= 0, \tilde{e}_1 \mathbf{Y}' = 0, \\ |Y'_1| &= |Y_1| = |Y_{s+1}| = |Y'_{s+1}|. \end{aligned}$$

Since  $\mathbf{Y} \neq \Phi$  and  $|Y'_1| = |Y_1|$ , we also have  $\mathbf{Y}' \neq \Phi$ . So by (4) either  $\tilde{e}_0 \mathbf{Y}' \neq 0$  or  $\tilde{e}_1 \mathbf{Y}' \neq 0$ , which is a contradiction. Hence  $|Y_1| \neq |Y_{s+1}|$ . ■

**Proposition 2** *Let  $\mathbf{Y} = (Y_1, \dots, Y_s, Y_{s+1}, \dots, Y_k) \in \mathcal{Y}(\Lambda)$ . Suppose that  $\tilde{f}_i^n \mathbf{Y} = \mathbf{Y}' = (Y'_1, \dots, Y'_s, Y'_{s+1}, \dots, Y'_k)$  for some  $n > 0$  and  $i = 0$  or 1. Then  $|Y'_j| \leq |Y_j| + 1$  for all  $1 \leq j \leq k$ .*

*Proof.* Let  $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$  be the  $i$ -signature of  $\mathbf{Y}$  and  $\varepsilon_J = (\varepsilon_{j_1}, \dots, \varepsilon_{j_r})$  be the relevant part of  $\varepsilon$ . By definition of  $\tilde{f}_i$  action (see rule C),  $\tilde{f}_i^n \mathbf{Y} = \mathbf{Y}' \neq 0$  implies that  $\varepsilon_{j_{r-n+1}} = \varepsilon_{j_{r-n+2}} = \dots = \varepsilon_{j_r} = 0$ . Also recall that each relevant 0 in  $\varepsilon_J$  corresponds to a unique concave  $i$ -corner in some  $Y_j$ ,  $1 \leq j \leq k$  and each application of  $\tilde{f}_i$  changes only one relevant 0 to a relevant 1 without affecting the partition of  $\varepsilon$ . Hence the first row of each  $Y'_j$  can have at most one more  $i$ -color node than that of  $Y_j$  for  $1 \leq j \leq k$ . Therefore, the proposition follows. ■

## 2 Crystals of Demazure modules

### 2.1 Demazure modules

The affine Lie algebra  $\widehat{sl}(2)$  has a triangular decomposition  $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ , where  $\mathfrak{h} = \text{span} \{h_0, h_1, d\}$  is the Cartan subalgebra and  $\mathfrak{n}^+$  (resp.  $\mathfrak{n}^-$ ) denote the sum of the positive (resp. negative) root spaces (see [18]). The subalgebra  $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^+$  is the Borel subalgebra of  $\widehat{sl}(2)$ . Let  $r_0, r_1$  denote the simple reflections corresponding to the simple roots  $\alpha_0, \alpha_1$  respectively. Recall that  $r_i \mu = \mu - \mu(h_i)\alpha_i$  for all  $\mu \in P$ . Let  $W$  denote the Weyl group of  $\widehat{sl}(2)$  generated by  $r_0, r_1$ . For  $w \in W$  let  $l(w)$  denote the length of  $w$  and let  $\prec$  denote the Bruhat order on  $W$ . For  $\Lambda = s\Lambda_0 + t\Lambda_1 \in P^+, k = s + t$ , as before we consider the integrable highest weight  $U_q(\widehat{sl}(2))$ -module  $V(\Lambda)$ . It is known that for  $w \in W$ , the extremal weight space  $V(\Lambda)_{w\Lambda}$  is one dimensional. Let  $E_w(\Lambda)$  denote the  $U_q(\mathfrak{b})$ -module generated by  $V(\Lambda)_{w\Lambda}$ . These modules  $E_w(\Lambda), w \in W$  are called the Demazure modules. They are finite-dimensional subspaces of  $V(\Lambda)$  and have the following property:

$$\begin{aligned} &\text{For } w, w' \in W, w \preceq w', \text{ we have } E_w(\Lambda) \subseteq E_{w'}(\Lambda), \\ &\text{and } \bigcup_{w \in W} E_w(\Lambda) = V(\Lambda). \end{aligned}$$

### 2.2 Demazure crystals

It is known that for  $\widehat{sl}(2)$ , the Weyl group is

$$W = \{(r_1 r_0)^m, r_0 (r_1 r_0)^m, (r_0 r_1)^m, r_1 (r_0 r_1)^m \mid m \in \mathbb{Z}_{\geq 0}\}.$$

In particular, for each integer  $L > 0$ , the Weyl group  $W$  has two distinct elements of length  $L$ :

$$w_L^+ = \underbrace{\cdots r_0 r_1 r_0}_L \quad \text{and} \quad w_L^- = \underbrace{\cdots r_1 r_0 r_1}_L.$$

Let  $W^+ = \{1, w_L^+ \mid L > 0\}$  and  $W^- = \{1, w_L^- \mid L > 0\}$ . Note that on each set  $W^+$  and  $W^-$  the Bruhat order is a total order. Also note that  $W^+ \cup W^- = W$  and  $W^+ \cap W^- = \{1\}$ . For  $\Lambda = s\Lambda_0 + t\Lambda_1 \in P^+$ , Kashiwara [10] has defined the crystals  $B_w(\Lambda)$  for the Demazure modules  $E_w(\Lambda)$  as a suitable subset of the crystal  $B(\Lambda)$ . For our purpose it suffices to recall the following recursive property of  $B_w(\Lambda)$ :

$$\begin{aligned} &\text{If } r_i w \prec w, \text{ then} \\ &B_w(\Lambda) = \{\tilde{f}_i^m b \mid m \geq 0, b \in B_{r_i w}(\Lambda), \tilde{e}_i b = 0\} \setminus \{0\}, \end{aligned} \tag{5}$$

which can be used to construct  $B_w(\Lambda)$  starting from the vacuum  $\Phi \in B(\Lambda)$ .

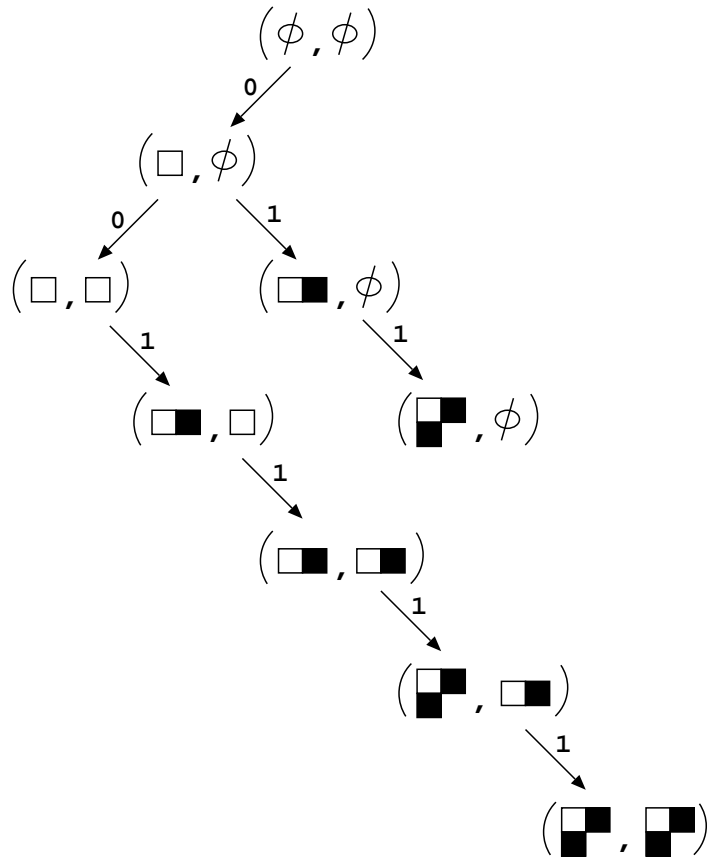


Figure 1: The crystal graph of  $B_{r_1 r_0}(2\Lambda_0)$ .

For example, if  $\Lambda = 2\Lambda_0$  and  $w = r_1 r_0$ , the graph of the crystal  $B_w(\Lambda)$  can be easily seen in Figure 1.

Since  $w_{L-1}^+ \prec w_L^+, w_{L-1}^- \prec w_L^-, w_{L-1}^+ \prec w_{L-1}^+ r_1 = w_L^-$  and  $w_{L-1}^- \prec w_{L-1}^- r_0 = w_L^+$ , the following result is an immediate consequence of Proposition 3.2.4 in [10].

**Proposition 3** *For  $L > 0$  and  $\Lambda \in P^+$  we have*

- (i)  $B_{w_{L-1}^+}(\Lambda) \subseteq B_{w_L^+}(\Lambda)$ ,
- (ii)  $B_{w_{L-1}^-}(\Lambda) \subseteq B_{w_L^-}(\Lambda)$ ,
- (iii)  $B_{w_{L-1}^+}(\Lambda) \subseteq B_{w_L^-}(\Lambda)$ ,
- (iv)  $B_{w_{L-1}^-}(\Lambda) \subseteq B_{w_L^+}(\Lambda)$ .

### 2.3 Realizations of $B_{w_L^+}(\Lambda)$ and $B_{w_L^-}(\Lambda)$

We define an extended Young diagram  $Y = (y_j)_{j \geq 0}$  of charge  $\gamma$  ( $=0$  or  $1$ ) to be *maximal of width  $L$*  if

$$|Y| = L \text{ and } y_{j+1} = y_j + 1 \text{ for } 0 \leq j \leq L-1.$$

We also define, for  $\Lambda = s\Lambda_0 + t\Lambda_1 \in P^+, k = s + t$ ,

$$B_L(\Lambda) = \{\mathbf{Y} = (Y_1, \dots, Y_s, Y_{s+1}, \dots, Y_k) \in B(\Lambda) \mid |Y_j| \leq L, 1 \leq j \leq k\}.$$

Noting that the weight multiplicities of  $w_L^+ \Lambda$  and  $w_L^- \Lambda$  are one, we define extremal vectors as follows:

**Definition 8 (extremal vectors)**  $b_{w_L^+ \Lambda}$  (resp.  $b_{w_L^- \Lambda}$ ) is defined as the element of  $B(\Lambda)$  which has the weight  $w_L^+ \Lambda$  (resp.  $w_L^- \Lambda$ ).

The following theorem characterizes the extremal vectors.

**Theorem 1** *Let  $\Lambda = s\Lambda_0 + t\Lambda_1, k = s + t$  and  $L > 0$ . Then we have*

- (i)  $b_{w_L^+ \Lambda} = (Y_1, \dots, Y_s, Y_{s+1}, \dots, Y_k) \in B(\Lambda)$  where  $Y_1 = \dots = Y_s$  are maximal of width  $L$  and charge  $0$ , and  $Y_{s+1} = \dots = Y_k$  are maximal of width  $L-1$  and charge  $1$ ,
- (ii)  $b_{w_L^- \Lambda} = (Y_1, \dots, Y_s, Y_{s+1}, \dots, Y_k) \in B(\Lambda)$  where  $Y_1 = \dots = Y_s$  are maximal of width  $L-1$  and charge  $0$ , and  $Y_{s+1} = \dots = Y_k$  are maximal of width  $L$  and charge  $1$ .

*Proof.* We use induction on  $L$ . If  $L = 1$ , then  $w_L^+ = r_0$  and  $w_L^+ \Lambda = r_0(s\Lambda_0 + t\Lambda_1) = \Lambda - s\alpha_0$ . Also it is easy to see that the 0-signature of the vacuum  $\Phi \in B(\Lambda)$  is  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_s)$  where  $\varepsilon_j = 0$  for  $1 \leq j \leq s$ . Hence

$$b_{r_0\Lambda} = \tilde{f}_0^s \Phi = (\square, \dots, \square, \phi, \dots, \phi) \in B(\Lambda)$$

and (i) holds for  $L = 1$ . Now assume (i) for  $L - 1$ . Observe that  $b_{w_{L-1}^+ \Lambda}$  has convex and concave corners of opposite colors. Furthermore,  $b_{w_{L-1}^+ \Lambda}$  has exactly  $sL + t(L - 1)$  number of concave  $i$ -corners where  $L - 1 \equiv i \pmod{2}$ . Also

$$\begin{aligned} w_L^+ \Lambda &= r_i(w_{L-1}^+ \Lambda) = w_{L-1}^+ \Lambda - (w_{L-1}^+ \Lambda)(h_i)\alpha_i \\ &= w_{L-1}^+ \Lambda - (sL + t(L - 1))\alpha_i. \end{aligned}$$

Since applying  $\tilde{f}_i$  to  $b_{w_{L-1}^+ \Lambda}$  ( $sL + t(L - 1)$ ) times we add one  $i$ -color box to each row of each diagram in  $b_{w_{L-1}^+ \Lambda}$ , we have

$$b_{w_L^+ \Lambda} = \tilde{f}_i^{sL+t(L-1)} b_{w_{L-1}^+ \Lambda} = (Y_1, \dots, Y_s, Y_{s+1}, \dots, Y_k) \in B(\Lambda)$$

where  $Y_1 = \dots = Y_s$  are maximal of width  $L$  and  $Y_{s+1} = \dots = Y_k$  are maximal of width  $L - 1$ . This proves (i). Proof of (ii) is similar to that of (i). ■

The following theorem characterizes the crystals of the Demazure modules.

**Theorem 2** *Let  $\Lambda = s\Lambda_0 + t\Lambda_1$ ,  $k = s + t$  and  $L > 0$ . Then we have*

$$\begin{aligned} B_{w_L^+}(\Lambda) &= \{(Y_1, \dots, Y_s, Y_{s+1}, \dots, Y_k) \in B(\Lambda) \mid |Y_1| \leq L, |Y_{s+1}| \leq L - 1\}, \\ B_{w_L^-}(\Lambda) &= \{(Y_1, \dots, Y_s, Y_{s+1}, \dots, Y_k) \in B(\Lambda) \mid |Y_1| \leq L - 1, |Y_{s+1}| \leq L\}. \end{aligned}$$

*Proof.* Let

$$\begin{aligned} S_L &= \{(Y_1, \dots, Y_s, Y_{s+1}, \dots, Y_k) \in B(\Lambda) \mid |Y_1| \leq L, |Y_{s+1}| \leq L - 1\}, \\ T_L &= \{(Y_1, \dots, Y_s, Y_{s+1}, \dots, Y_k) \in B(\Lambda) \mid |Y_1| \leq L - 1, |Y_{s+1}| \leq L\}. \end{aligned}$$

We use induction on  $L$ . First we will show that  $B_{w_L^+}(\Lambda) \subseteq S_L$  and  $B_{w_L^-}(\Lambda) \subseteq T_L$ . This is clear for  $L = 1$  since by (5)  $B_{r_0}(\Lambda) = \{\tilde{f}_0^m \Phi \mid 0 \leq m \leq s\}$  and  $B_{r_1}(\Lambda) = \{\tilde{f}_1^m \Phi \mid 0 \leq m \leq t\}$ . Assume that  $B_{w_{L-1}^+}(\Lambda) \subseteq S_{L-1}$  and  $B_{w_{L-1}^-}(\Lambda) \subseteq T_{L-1}$ . Since  $w_L^+ = r_i w_{L-1}^+$  where  $L - 1 \equiv i \pmod{2}$  by (5) we have

$$B_{w_L^+}(\Lambda) = \{\tilde{f}_i^m b \mid m \geq 0, b \in B_{w_{L-1}^+}(\Lambda), \tilde{e}_i b = 0\} \setminus \{0\}.$$

Since  $B_{w_{L-1}^+}(\Lambda) \subseteq S_{L-1}$ , by Proposition 2 we have  $B_{w_L^+}(\Lambda) \subseteq S_L$ . Similarly  $B_{w_L^-}(\Lambda) \subseteq T_L$ .

Now we proceed to show that  $S_L \subseteq B_{w_L^+}(\Lambda)$  and  $T_L \subseteq B_{w_L^-}(\Lambda)$ . As before this is clear for  $L = 1$ . Assume that  $S_{L-1} \subseteq B_{w_{L-1}^+}(\Lambda)$  and  $T_{L-1} \subseteq B_{w_{L-1}^-}(\Lambda)$ .

Let  $\mathbf{Y} = (Y_1, \dots, Y_s, Y_{s+1}, \dots, Y_k) \in S_L$ . By Proposition 1,  $|Y_1| \neq |Y_{s+1}|$ . If  $|Y_1| \leq L-1$  and  $|Y_{s+1}| \leq L-2$ , then  $\mathbf{Y} \in S_{L-1} \subseteq B_{w_{L-1}^+}(\Lambda) \subseteq B_{w_L^+}(\Lambda)$  by Proposition 3(i). If  $|Y_1| \leq L-2$  and  $|Y_{s+1}| \leq L-1$ , then  $\mathbf{Y} \in T_{L-1} \subseteq B_{w_{L-1}^-}(\Lambda) \subseteq B_{w_L^+}(\Lambda)$  by Proposition 3(iv).

Finally, if  $|Y_1| = L$  and  $|Y_{s+1}| \leq L-1$ , then  $Y_1$  will have a convex  $i$ -corner corresponding to the last node of the first row of  $Y_1$  where  $L-1 \equiv i \pmod{2}$ . Furthermore, this convex  $i$ -corner will contribute a relevant 1 to the  $i$ -signature of  $\mathbf{Y}$ . Therefore,  $\tilde{e}_i \mathbf{Y} \neq 0$ . Now, we can choose  $m$  such that  $\tilde{e}_i^m \mathbf{Y} = \mathbf{Y}' = (Y'_1, \dots, Y'_s, Y'_{s+1}, \dots, Y'_k) \in B(\Lambda)$  and  $\tilde{e}_i \mathbf{Y}' = 0$ . Then  $|Y'_1| = L-1$ , and  $|Y'_{s+1}| \leq L-2$  by Proposition 1. Thus  $\mathbf{Y}' \in S_{L-1} \subseteq B_{w_{L-1}^+}(\Lambda)$ . Hence by (5)  $\mathbf{Y} = \tilde{f}_i^m \mathbf{Y}' \in B_{r_i w_{L-1}^+}(\Lambda) = B_{w_L^+}(\Lambda)$ . This proves that  $S_L \subseteq B_{w_L^+}(\Lambda)$ . Similarly  $T_L \subseteq B_{w_L^-}(\Lambda)$  and the theorem holds. ■

The following result is an immediate consequence of Theorem 2.

**Corollary 1** For  $\Lambda = s\Lambda_0 + t\Lambda_1$ , we have

$$\begin{aligned} (i) \quad & B_{w_L^+}(\Lambda) \cup B_{w_L^-}(\Lambda) = B_L(\Lambda), \\ (ii) \quad & B_{w_L^+}(\Lambda) \cap B_{w_L^-}(\Lambda) = B_{L-1}(\Lambda). \end{aligned}$$

Let us consider the extreme cases  $s = 0$  or  $t = 0$ . Assume  $\Lambda = k\Lambda_0$ . Noting  $w_L^- \Lambda = w_{L-1}^+ \Lambda$  we have  $B_{w_L^-}(\Lambda) = B_{w_{L-1}^+}(\Lambda)$ . Similarly, if  $\Lambda = k\Lambda_1$ , we have  $B_{w_L^+}(\Lambda) = B_{w_{L-1}^-}(\Lambda)$ . Combining Proposition 3 and the previous corollary, we have

**Corollary 2**

$$B_L(\Lambda) = \begin{cases} B_{w_L^+}(\Lambda) & \text{if } \Lambda = k\Lambda_0, \\ B_{w_L^-}(\Lambda) & \text{if } \Lambda = k\Lambda_1. \end{cases}$$

## 3 Characters

### 3.1 1D configuration sum

We summarize the 1D configuration sum which have already appeared in the study of exactly solvable SOS models [15]. It will be used in the evaluation of the character for  $\mathcal{P}_L(\Lambda)$  in the following subsection.

Fixing a positive integer  $k$ , let us call a pair of intergers  $(b, c)$  *weakly admissible* if  $b - c = -k, -k+2, \dots, k$ . For a weakly admissible pair  $(b, c)$  and  $L \in \mathbb{Z}_{\geq 0}$ , let  $f_L^{(k)}(b, c)$  denote the unique solution to the following linear difference equation.

$$f_L^{(k)}(b, c) = \sum_d' f_{L-1}^{(k)}(d, b) q^{L|d-c|/4}, \quad (6)$$

$$f_0^{(k)}(b, c) = \delta_{b0}. \quad (7)$$

Here the sum  $\sum'$  is taken over  $d$  such that the pair  $(d, b)$  is weakly admissible. If  $(b, c)$  is not weakly admissible,  $f_L^{(k)}(b, c)$  is set to be 0. Note that  $N, m$  in [15] are replaced with  $k, L$ , respectively.  $f_L^{(k)}(b, c)$  enjoys the following properties.

Reflection symmetry:

$$f_L^{(k)}(b, c) = f_L^{(k)}(-b, -c),$$

Support property:

$$f_L^{(k)}(b, c) = 0 \quad \text{unless } |b| \leq Lk, |c| \leq (L+1)k \text{ and } b \equiv Lk \pmod{2}.$$

Given an integer  $\mu$  set

$$R_\mu = \{b \in \mathbb{Z} \mid (\mu - 1)k \leq b \leq (\mu + 1)k\},$$

where the left (resp. right) equality sign is taken if  $\mu - 1 \leq 0$  (resp.  $\mu + 1 \geq 0$ ). We also set  $R_{\mu, \nu} = R_\mu \times R_\nu$ . For a weakly admissible pair  $(b, c)$  and integer  $L(\geq 1)$ , let  $\mu, \nu (= \mu \pm 1)$  be integers such that

$$(b, c) \in R_{\mu, \nu}, \quad \mu \equiv L + 1, \nu \equiv L + 2 \pmod{2}.$$

The above equation uniquely determines  $\mu$  and  $\nu$  except when  $b = 0(\mu = \pm 1)$  or  $c = 0(\nu = \pm 1)$ , in which cases either choice is allowed.

Let us recall standard notations in  $q$  analysis (see [21] for example). We set

$$(z; q)_m = \begin{cases} \prod_{j=1}^m (1 - zq^{j-1}) & (m \geq 1), \\ 1 & (m = 0). \end{cases}$$

If there is no danger of confusion, we abbreviate it as  $(z)_m$ . The  $q$ -multinomial coefficient is defined as

$$\begin{bmatrix} M \\ m_1 \cdots m_n \end{bmatrix}_q = \begin{cases} \frac{(q)_M}{\prod_{j=1}^n (q)_{m_j}} & \text{if } m_1, \dots, m_n \geq 0 \\ & \text{and } m_1 + \dots + m_n = M, \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

$\begin{bmatrix} M \\ m_1 m_2 \end{bmatrix}_q$  is often written as  $\begin{bmatrix} M \\ m_1 \end{bmatrix}_q$  and called the *Gaussian polynomial*. In both cases, if the subscript is omitted, we understand it as  $q$ .

For later use we extend the definition of the Gaussian polynomial  $\begin{bmatrix} M \\ i \end{bmatrix}$  with  $M \geq 0, i \in \mathbb{Z}$  to the case of  $M, i \in \mathbb{Z}$ , by setting

$$\begin{bmatrix} M \\ i \end{bmatrix} = \begin{cases} \frac{(q^{M-i+1})_i}{(q)_i} & \text{if } i \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Similarly, for  $M < 0$  we define  $(z)_M$  to be  $(q^M z)^{-1}_{-M}$ .

**Lemma 2** For  $M, N, n \in \mathbb{Z}$  we have

$$(z)_M = \sum_i (-z)^i q^{i(i-1)/2} \begin{bmatrix} M \\ i \end{bmatrix}, \quad (10)$$

$$\begin{bmatrix} M+N \\ n \end{bmatrix} = \sum_i q^{(n-i)(M-i)} \begin{bmatrix} M \\ i \end{bmatrix} \begin{bmatrix} N \\ n-i \end{bmatrix}, \quad (11)$$

where the sums are taken over all integers.

*Proof.* The first equation reduces to (3.3.6) or (3.3.7) in [21] depending on whether  $M \geq 0$  or  $M < 0$ .

Next, note that the following equation holds for  $M, N \in \mathbb{Z}$ :

$$(z)_{M+N} = (z)_M (q^M z)_N. \quad (12)$$

The second equation can be proved by using (10) in the both sides of (12) and comparing the coefficients of  $z^n$ . ■

We call the following expression bosonic.

**Proposition 4 (Theorem 4.1.1 in [15])** For any weakly admissible pair  $(b, c) \in R_{\mu, \nu}$  and integers  $L, k \geq 1$ ,

$$\begin{aligned} & (q)_{L-1} f_L^{(k)}(b, c) \\ &= \left( \sum_{\substack{i \geq (L+\nu)/2 \\ j \leq (L+\mu-1)/2}} - \sum_{\substack{i \leq (L+\nu)/2-1 \\ j \geq (L+\mu+1)/2}} \right) (-1)^{i+j} q^{Q_{i,j}^{(L)}(b,c)} \begin{bmatrix} L-1 \\ i \end{bmatrix} \begin{bmatrix} L \\ j \end{bmatrix}, \quad (13) \\ & Q_{i,j}^{(L)}(b, c) \\ &= \frac{1}{2}(i-j)(i-j+1) - \left(i - \frac{L-1}{2}\right) \left(j - \frac{L}{2}\right) k \\ & \quad + \frac{b}{2} \left(i - \frac{L-1}{2}\right) + \frac{c}{2} \left(j - \frac{L}{2}\right). \end{aligned}$$

### 3.2 Fermionic expression

In this subsection we rewrite the bosonic expression to the fermionic expression which involves  $q$ -multinomials. At the first step we show the following.

**Proposition 5** For any weakly admissible pair  $(b, c) \in R_{\mu, \nu}$  satisfying  $b \geq -Lk$  and integers  $L, k \geq 1$ ,

$$\begin{aligned} & q^{\frac{L(L-1)k + (L-1)b + Lc}{4}} f_L^{(k)}(b, c) \\ &= \sum_{j \in \mathbb{Z}} (-1)^j q^{\frac{1}{2}j(j-1) + \frac{L-1}{2}jk + \frac{c}{2}j} \begin{bmatrix} L \\ j \end{bmatrix} \begin{bmatrix} L-j-1 - (j - \frac{L}{2})k + \frac{b}{2} \\ L-1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
& - \sum_{j \in \mathbb{Z}} (-1)^j q^{\frac{1}{2}j(j+1) + \frac{L}{2}jk + \frac{b}{2}j} \begin{bmatrix} L-1 \\ j \end{bmatrix} \begin{bmatrix} L-j-2 - (j - \frac{L-1}{2})k + \frac{c}{2} \\ L-1 \end{bmatrix} \\
& \quad \times \left( 1 - q^{L-j-1 - (j - \frac{L-1}{2})k + \frac{c}{2}} \right).
\end{aligned}$$

*Proof.* Rewrite the sum

$$\left( \sum_{\substack{i \geq (L+\nu)/2 \\ j \leq (L+\mu-1)/2}} - \sum_{\substack{i \leq (L+\nu)/2-1 \\ j \geq (L+\mu+1)/2}} \right)$$

in (13) as

$$\left( \sum_{\substack{i \in \mathbb{Z} \\ j \leq (L+\mu-1)/2}} - \sum_{\substack{i \leq (L+\nu)/2-1 \\ j \in \mathbb{Z}}} \right).$$

Applying (10) for the sum over  $i \in \mathbb{Z}$  or  $j \in \mathbb{Z}$ , we obtain

$$\begin{aligned}
& q^{\frac{L(L-1)k + (L-1)b + Lc}{4}} f_L^{(k)}(b, c) \\
& = \sum_{j \leq \frac{L+\mu-1}{2}} (-1)^j q^{\frac{1}{2}j(j-1) + \frac{L-1}{2}jk + \frac{c}{2}j} \begin{bmatrix} L \\ j \end{bmatrix} \frac{(q^{-j+1 - (j - \frac{L}{2})k + \frac{b}{2}})_{L-1}}{(q)_{L-1}} \\
& \quad - \sum_{i \leq \frac{L+\nu}{2}-1} (-1)^i q^{\frac{1}{2}i(i+1) + \frac{L}{2}ik + \frac{b}{2}i} \begin{bmatrix} L-1 \\ i \end{bmatrix} \frac{(q^{-i - (i - \frac{L-1}{2})k + \frac{c}{2}})_L}{(q)_{L-1}}.
\end{aligned}$$

Recall that we have assumed  $b \geq -Lk$ . Noticing  $(\mu-1)k \leq b \leq (\mu+1)k$ ,  $(\nu-1)k \leq c \leq (\nu+1)k$  with  $\mu \geq 1-L$ ,  $\nu \geq -L$ , we get  $L-j-1 - (j - \frac{L}{2})k + \frac{b}{2} < m-1$  and  $L-i-2 - (i - \frac{L-1}{2})k + \frac{c}{2} < m-1$  if  $j > \frac{L+\mu-1}{2}$  and  $i > \frac{L+\nu}{2}-1$ . Therefore, recalling the definition (9), we have

$$\begin{aligned}
\begin{bmatrix} L-j-1 - (j - \frac{L}{2})k + \frac{b}{2} \\ L-1 \end{bmatrix} &= \begin{cases} \frac{(q^{-j+1 - (j - \frac{L}{2})k + \frac{b}{2}})_{L-1}}{(q)_{L-1}} & \text{if } j \leq \frac{L+\mu-1}{2} \\ 0 & \text{otherwise,} \end{cases} \\
\begin{bmatrix} L-i-2 - (i - \frac{L-1}{2})k + \frac{c}{2} \\ L-1 \end{bmatrix} &= \begin{cases} \frac{(q^{-i - (i - \frac{L-1}{2})k + \frac{c}{2}})_L}{(q)_{L-1}} & \text{if } i \leq \frac{L+\nu}{2}-1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus we arrive at the desired result. ■

**Proposition 6** For any weakly admissible pair  $(b, c) \in R_{\mu, \nu}$  satisfying  $b \geq 0$ ,  $c \neq b+k$  and integers  $L \geq 0$ ,  $k \geq 2$ , we have

$$\begin{aligned}
f_L^{(k)}(b, c) &= \sum_{0 \leq i \leq L} q^{\frac{L(L-1)}{4} - \frac{(k-1)i^2 + (2L+b+c-1)i}{4}} \begin{bmatrix} L \\ i \end{bmatrix} \\
& \quad \times f_{L-i}^{(k-1)}(b + (k+1)i - L, c + (k+1)i - L + 1).
\end{aligned}$$

*Proof.* If  $L = 0$ , the formula is shown easily from the initial condition (7). We assume  $L \geq 1$ .

Notice that from the assumption  $f_{L-i}^{(k-1)}(b+(k+1)i-L, c+(k+1)i-L+1) = 0$  when  $i = L$ . Since the right hand side of the formula in Proposition 5 is formally zero when  $L = 0$ , we can use it for convenience of our proof. Check that the assumption allows us to use Proposition 5. Setting

$$\begin{aligned} a_L^{(k)}(b, c; j) &= q^{-\frac{L(L-1)k+(L-1)b+Lc}{4} + \frac{L-1}{2}jk + \frac{c}{2}j} \begin{bmatrix} L \\ j \end{bmatrix} \begin{bmatrix} L-j-1-(j-\frac{L}{2})k+\frac{b}{2} \\ L-1 \end{bmatrix}, \\ b_L^{(k)}(b, c; j) &= q^{-\frac{L(L-1)k+(L-1)b+Lc}{4} + \frac{L}{2}jk + \frac{b}{2}j} \begin{bmatrix} L-1 \\ j \end{bmatrix} \\ &\quad \times \begin{bmatrix} L-j-2-(j-\frac{L-1}{2})k+\frac{c}{2} \\ L-1 \end{bmatrix} \left(1 - q^{L-j-1-(j-\frac{L-1}{2})k+\frac{c}{2}}\right), \end{aligned}$$

we have to show

$$\begin{aligned} &\sum_{j \in \mathbb{Z}} (-1)^j \left( q^{\frac{1}{2}j(j-1)} a_L^{(k)}(b, c; j) - q^{\frac{1}{2}j(j+1)} b_L^{(k)}(b, c; j) \right) \\ &= \sum_{i=0}^L q^{\frac{L(L-1)}{4} - \frac{(k-1)i^2 + (2L+b+c-1)i}{4}} \begin{bmatrix} L \\ i \end{bmatrix} \\ &\quad \times \sum_{j \in \mathbb{Z}} (-1)^j \left( q^{\frac{1}{2}j(j-1)} a_{L-i}^{(k-1)}(b', c'; j) - q^{\frac{1}{2}j(j+1)} b_{L-i}^{(k-1)}(b', c'; j) \right), \end{aligned}$$

where  $b' = b + (k+1)i - L$ ,  $c' = c + (k+1)i - L + 1$ . In fact, we can show that the recurrence relation holds term by term, that is,

$$\begin{aligned} a_L^{(k)}(b, c; j) &= \sum_{i=0}^L q^{\frac{L(L-1)}{4} - \frac{(k-1)i^2 + (2L+b+c-1)i}{4}} \begin{bmatrix} L \\ i \end{bmatrix} \\ &\quad \times a_{L-i}^{(k-1)}(b + (k+1)i - L, c + (k+1)i - L + 1; j), \quad (14) \end{aligned}$$

and the same for  $b_L^{(k)}(b, c; j)$ . After some calculation, (14) reduces to

$$\begin{aligned} &\begin{bmatrix} L \\ j \end{bmatrix} \begin{bmatrix} L-j-1-(j-\frac{L}{2})k+\frac{b}{2} \\ L-1 \end{bmatrix} \\ &= \sum_i q^{(L-i-1)(L-i-j)} \begin{bmatrix} L \\ i \end{bmatrix} \begin{bmatrix} L-i \\ j \end{bmatrix} \begin{bmatrix} -1-(j-\frac{L}{2})k+\frac{b}{2} \\ L-i-1 \end{bmatrix}. \end{aligned}$$

Noting that  $\begin{bmatrix} L \\ i \end{bmatrix} \begin{bmatrix} L-i \\ j \end{bmatrix} = \begin{bmatrix} L \\ j \end{bmatrix} \begin{bmatrix} L-j \\ i \end{bmatrix}$ , we can cancel the factor  $\begin{bmatrix} L \\ j \end{bmatrix}$ . Thus we arrive at (11) with  $M = L-j$ ,  $N = -1 - (j - \frac{L}{2})k + \frac{b}{2}$ ,  $n = L-1$ .

Similarly, (14) with  $a_L^{(k)}(b, c; j)$  replaced by  $b_L^{(k)}(b, c; j)$  reduces to

$$\begin{bmatrix} L-1 \\ j \end{bmatrix} \begin{bmatrix} L-j-2-(j-\frac{L-1}{2})k+\frac{c}{2} \\ L-1 \end{bmatrix} \left(1 - q^{L-j-1-(j-\frac{L-1}{2})k+\frac{c}{2}}\right)$$

$$\begin{aligned}
&= \sum_i q^{(L-i)(L-i-j-1)} \begin{bmatrix} L \\ i \end{bmatrix} \begin{bmatrix} L-i-1 \\ j \end{bmatrix} \begin{bmatrix} -1 - (j - \frac{L-1}{2})k + \frac{c}{2} \\ L-i-1 \end{bmatrix} \\
&\quad \times \left(1 - q^{-(j - \frac{L-1}{2})k + \frac{c}{2}}\right). \quad (15)
\end{aligned}$$

Set  $N = -(j - \frac{L-1}{2})k + \frac{c}{2}$ . Noting that

$$\begin{aligned}
&\begin{bmatrix} L-1 \\ j \end{bmatrix} \begin{bmatrix} L-j+N-2 \\ L-1 \end{bmatrix} (1 - q^{L-j+N-1}) \\
&= \begin{bmatrix} L \\ j \end{bmatrix} \begin{bmatrix} L-j+N-1 \\ L \end{bmatrix} (1 - q^{L-j}), \\
&\begin{bmatrix} L \\ i \end{bmatrix} \begin{bmatrix} L-i-1 \\ j \end{bmatrix} \begin{bmatrix} N-1 \\ L-i-1 \end{bmatrix} (1 - q^N) \\
&= \begin{bmatrix} L \\ j \end{bmatrix} \begin{bmatrix} N \\ L-i \end{bmatrix} \begin{bmatrix} L-j-1 \\ i \end{bmatrix} (1 - q^{L-j}),
\end{aligned}$$

and cancelling the factor  $\begin{bmatrix} L \\ j \end{bmatrix} (1 - q^{L-j})$ , (15) reduces again to (11) with  $M = L - j - 1, n = L$ . ■

This recurrence relation with respect to  $k$  leads us to an expression of  $f_L^{(k)}(b, c)$  in terms of the  $q$ -multinomial coefficient (8). Next theorem is a generalization of Theorem 4.2.5 in [15].

**Theorem 3** *For any weakly admissible pair  $(b, c) \in R_{\mu, \nu}$  and integers  $L \geq 0, k \geq 1$ , we have the following expression for  $f_L^{(k)}(b, c)$ .*

$$\begin{aligned}
f_L^{(k)}(b, c) &= \sum q^{\mathcal{Q}} \begin{bmatrix} L \\ x_0 \cdots x_k \end{bmatrix}, \quad (16) \\
\mathcal{Q} &= \frac{L^2 k - L(b - c + k) + b}{4} - \sum_{0 \leq a \leq a' \leq k} (a' - a) x_a x_{a'} \\
&\quad + \sum_{\frac{c-b+k}{2} \leq a \leq k} (a - \frac{c-b+k}{2}) x_a.
\end{aligned}$$

The sum in (16) is taken over all non negative integers  $x_0, \dots, x_k$  satisfying

$$\sum_{a=0}^k x_a = L, \quad \sum_{a=0}^k a x_a = \frac{Lk - b}{2}.$$

*Proof.* We prove by induction on  $k$ . If  $k = 1$ , (16) reduces to

$$f_L^{(1)}(b, c) = q^{\frac{bc}{4}} \begin{bmatrix} L \\ \frac{L+b}{2} \frac{L-b}{2} \end{bmatrix}.$$

This trivially satisfies (7). Checking (6) reduces to the following identity of Gaussian polynomials.

$$\begin{bmatrix} M \\ i \end{bmatrix} = \begin{bmatrix} M-1 \\ i-1 \end{bmatrix} + q^i \begin{bmatrix} M-1 \\ i \end{bmatrix} = q^{M-i} \begin{bmatrix} M-1 \\ i-1 \end{bmatrix} + \begin{bmatrix} M-1 \\ i \end{bmatrix}.$$

Thus the theorem is correct when  $k = 1$ .

Now assume  $k \geq 2$ . If  $c = b + k$ , (16) turns out to be equivalent to Theorem 4.2.5 (see also equation (4.2.1)) in [15]. Therefore we can assume  $c \neq b + k$ . First suppose  $b \geq 0$ . From Proposition 6 and the assumption of the induction, we obtain

$$f_L^{(k)}(b, c) = \sum_{x_k=0}^L \sum q^{\mathcal{Q}} \begin{bmatrix} L \\ x_k \end{bmatrix} \begin{bmatrix} L - x_k \\ x_0 \cdots x_{k-1} \end{bmatrix}, \quad (17)$$

with

$$\begin{aligned} \mathcal{Q} &= \frac{L(L-1)}{4} - \frac{(k-1)x_k^2 + (2L+b+c-1)x_k}{4} + \mathcal{Q}', \\ \mathcal{Q}' &= \frac{(L-x_k)^2(k-1) - (L-x_k)(b'-c'+k-1) + b'}{4} \\ &\quad - \sum_{0 \leq a < a' \leq k-1} (a' - a)x_a x_{a'} + \sum_{\frac{c'-b'+k-1}{2} \leq a \leq k-1} (a - \frac{c'-b'+k-1}{2})x_a, \\ b' &= b + (k+1)x_k - L, \quad c' = c + (k+1)x_k - L + 1, \end{aligned}$$

and the inner sum in (17) is taken over non negative integers  $x_0, \dots, x_{k-1}$  such that  $\sum_{a=0}^{k-1} x_a = L - x_k$ ,  $\sum_{a=0}^{k-1} ax_a = \frac{(L-x_k)(k-1)-b'}{2}$ . Calculating the power of  $q$  and noting that

$$\begin{bmatrix} L \\ x_k \end{bmatrix} \begin{bmatrix} L - x_k \\ x_0 \cdots x_{k-1} \end{bmatrix} = \begin{bmatrix} L \\ x_0 \cdots x_k \end{bmatrix},$$

we arrive at (16).

For the proof of the case  $b < 0$ , it suffices to check Reflection symmetry in the right hand side of (16). The check is easily done by the variable change  $x_j \rightarrow x_{k-j}$  ( $0 \leq j \leq k$ ) in the summand of (16). ■

### 3.3 Character of $\mathcal{P}_L(\Lambda)$

We define the character for  $\mathcal{P}_L(\Lambda)$ .

**Definition 9 (character of  $\mathcal{P}_L(\Lambda)$ )**

$$ch_L(\Lambda) = ch_L(\Lambda)(z, q) = \sum_{p \in \mathcal{P}_L(\Lambda)} z^{(\Lambda - p_0 | \Lambda_1)} q^{E(p)}.$$

Then we have the following expression for  $ch_L(\Lambda)$ .

**Proposition 7**

$$ch_L(\Lambda) = \sum_{j \in \mathbb{Z}} z^{-j} q^{j/2} f_L^{(k)}(\epsilon^{(L)}(s-t) - 2j, \epsilon^{(L+1)}(s-t) - 2j).$$

Here  $\epsilon^{(L)}$  is given in (2).

*Proof.* First, for  $a\Lambda_0 + b\Lambda_1 \in P_k$ , we set  $\nu(a\Lambda_0 + b\Lambda_1) = b$ . Then we have

$$\mu(\iota(p)_j) = \frac{\nu(p_{j+1}) - \nu(p_j) + k}{2}.$$

From the definition of  $H$  (Definition 5), we have

$$\begin{aligned} H(\iota(p)_{j-1}, \iota(p)_j) &= \frac{k}{2} + \frac{|\nu(p_{j-1}) - \nu(p_{j+1})|}{4} \\ &\quad + \frac{\nu(p_{j-1}) - 2\nu(p_j) + \nu(p_{j+1})}{4}, \end{aligned}$$

and therefore,

$$\begin{aligned} \sum_{j=1}^L j H(\iota(p)_{j-1}, \iota(p)_j) &= \frac{k}{2} \frac{L(L+1)}{2} + \sum_{j=1}^L j \frac{|\nu(p_{j-1}) - \nu(p_{j+1})|}{4} \\ &\quad + \frac{\nu(p_0) - (L+1)\nu(p_L) + L\nu(p_{L+1})}{4}. \end{aligned}$$

Thus we have

$$E(p) = \frac{\nu(p_0) - \nu(\bar{p}_0)}{4} + \sum_{j=1}^L j \frac{|\nu(p_{j-1}) - \nu(p_j)|}{4}.$$

Using this and parametrizing  $p_0$  by  $\Lambda + j\alpha_1$  ( $j \in \mathbb{Z}$ ), we can rewrite the character (Definition 9) as

$$\begin{aligned} ch_L(\Lambda) &= \sum_{j \in \mathbb{Z}} z^{-j} q^{(\nu(p_0) - \nu(\bar{p}_0))/4} K_{L,j}, \\ K_{L,j} &= \sum_{\substack{p \in \mathcal{P}_L(\Lambda) \\ p_0 = \Lambda + j\alpha_1}} q^{\sum_{j=1}^L j |\nu(p_{j-1}) - \nu(p_{j+1})|/4}. \end{aligned}$$

Noting that  $K_{L,j}$  satisfies the same difference equation as  $f_L^{(k)}(\nu(p_L) - \nu(p_0), \nu(p_{L+1}) - \nu(p_0))$ , we have

$$ch_L(\Lambda) = \sum_{j \in \mathbb{Z}} z^{-j} q^{(\nu(p_0) - \nu(\bar{p}_0))/4} f_L^{(k)}(\nu(p_L) - \nu(p_0), \nu(p_{L+1}) - \nu(p_0))$$

with  $p_0 = \Lambda + j\alpha_1$ . Evaluating  $\nu(p_j)$  ( $j = 0, L, L+1$ ) explicitly, we have the desired result. ■

Now we have the final result on the character of  $\mathcal{P}_L(\Lambda)$ .

**Theorem 4** Let  $\Lambda = s\Lambda_0 + t\Lambda_1$  ( $s + t = k$ ),  $C$  be the Cartan matrix of  $sl(k)$ , that is,  $C_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}$  ( $1 \leq i, j \leq k-1$ ). Set  $x^t = (x_1, \dots, x_{k-1})$  and let  $e_i$  be the  $k-1$  dimensional  $i$ -th unit vector ( $e_0 = e_k = 0$ ). Define  $F_{j,L}(\Lambda)$  by

$$ch_L(\Lambda) = \sum_{j \in \mathbb{Z}} z^{-j} F_{j,L}(\Lambda).$$

Then we have the following expression for  $F_{j,L}(\Lambda)$ .  
If  $L$  is even,

$$F_{j,L}(\Lambda) = \sum_{\sum_{i=1}^{k-1} ix_i \in j+k\mathbb{Z}} q^{x^t C^{-1} x - x^t C^{-1} e_s + \frac{j(j+t)}{k}} \begin{bmatrix} L \\ x_0 \cdots x_k \end{bmatrix},$$

where

$$x_0 = \frac{L}{2} - \frac{1}{k} \left( \sum_{i=1}^{k-1} (k-i)x_i + j \right), \quad x_k = \frac{L}{2} - \frac{1}{k} \left( \sum_{i=1}^{k-1} ix_i - j \right).$$

If  $L$  is odd,

$$F_{j,L}(\Lambda) = \sum_{\sum_{i=1}^{k-1} ix_i \in j+t+k\mathbb{Z}} q^{x^t C^{-1} x - x^t C^{-1} e_t + \frac{j(j+t)}{k}} \begin{bmatrix} L \\ x_0 \cdots x_k \end{bmatrix},$$

where

$$x_0 = \frac{L+1}{2} - \frac{1}{k} \left( \sum_{i=1}^{k-1} (k-i)x_i + j + t \right), \quad x_k = \frac{L-1}{2} - \frac{1}{k} \left( \sum_{i=1}^{k-1} ix_i - j - t \right).$$

*Proof.* Suppose  $L$  is even, from Theorem 3 and Proposition 7, we obtain

$$F_{j,L}(\Lambda) = \sum_{\sum_{i=0}^k ix_i = \frac{Lk}{2} + j} q^{\frac{L^2 k}{4} - \frac{Lt}{2} - \sum_{0 \leq i \leq i' \leq k} (i' - i)x_i x_{i'} + \sum_{s \leq i \leq k} (i - s)x_i} \begin{bmatrix} L \\ x_0 \cdots x_k \end{bmatrix}.$$

Solving  $\sum_{i=0}^k x_i = L$  and  $\sum_{i=0}^k ix_i = \frac{Lk}{2} + j$  for  $x_0$  and  $x_k$ , we get  $x_0 = \frac{L}{2} - \frac{1}{k} \left( \sum_{i=1}^{k-1} (k-i)x_i + j \right)$ ,  $x_k = \frac{L}{2} - \frac{1}{k} \left( \sum_{i=1}^{k-1} ix_i - j \right)$ . Substitute these expressions into the power of  $q$ , and note that  $(C^{-1})_{ij} = \frac{i(k-j)}{k}$  (for  $i \leq j$ ),  $\frac{j(k-i)}{k}$  (for  $i > j$ ). Noticing that the sum restriction changes into  $\sum_{i=1}^{k-1} ix_i \in j + k\mathbb{Z}$ , we obtain the desired expression.

The case of  $L$  odd is similar. ■

### 3.4 Character of Demazure module

As we have seen in section 2, crystals of Demazure modules can be realized as subsets of the crystal  $B(\Lambda)$  or equivalently  $\mathcal{P}(\Lambda)$ . Now it is easy to see that the characters of Demazure modules admit the following definition.

**Definition 10**

$$ch_L^\pm(\Lambda) = ch_L^\pm(\Lambda)(z, q) = \sum_{p \in B_{w_L^\pm}(\Lambda)} z^{(\Lambda - p_0 | \Lambda_1)} q^{E(p)}.$$

A natural question to ask is: Is it possible to express  $ch_L^\pm(\Lambda)$  in terms of  $ch_L(\Lambda)$ ? The answer is given by the following theorem.

**Theorem 5** *Let  $\Lambda = s\Lambda_0 + t\Lambda_1$ ,  $s + t = k$ . We have*

$$\begin{aligned} ch_L^+(\Lambda) &= \sum_{i=0}^s z^{-\epsilon^{(L)}(s-i)} q^{\frac{L+\epsilon^{(L)}}{2}(s-i)} ch_{L-1}(\Lambda^{(i)}), \\ ch_L^-(\Lambda) &= \sum_{i=0}^t z^{\epsilon^{(L)}(t-i)} q^{\frac{L-\epsilon^{(L)}}{2}(t-i)} ch_{L-1}(\Lambda^{(k-i)}). \end{aligned}$$

Here  $\epsilon^{(L)}$  is given in (2) and  $\Lambda^{(i)} = i\Lambda_0 + (k-i)\Lambda_1$ .

*Proof.* First we prove the expression for  $ch_L^+(\Lambda)$ . Let us suppose  $L$  is even. Interpreting Theorem 2 in terms of paths, for any  $p \in B_{w_L^+}(\Lambda)$  we have  $\iota(p)_{L-1} = (k-i)\epsilon_0 + i\epsilon_1$  where  $i = 0, 1, \dots, s$ . For each  $i$  consider the following subset of  $\mathcal{P}_L(\Lambda)$ .

$$S^{(i)} = \{p \in \mathcal{P}_L(\Lambda) \mid p_{L-1} = \Lambda - ((k-i)\widehat{0} + i\widehat{1})\}.$$

Noting that  $\Lambda - ((k-i)\widehat{0} + i\widehat{1}) - \Lambda^{(k-i)} = \Lambda - \Lambda^{(i)}$ , we see that

$$\{(p_j - (\Lambda - \Lambda^{(i)}))_{j=0, \dots, L} \mid p \in S^{(i)}\}$$

can be identified with  $\mathcal{P}_{L-1}(\Lambda^{(i)})$ . Defining  $\bar{p}^{(i)}$  as the ground-state path of  $\mathcal{P}_{L-1}(\Lambda^{(i)})$ , we have

$$\begin{aligned} ch_{L-1}(\Lambda^{(i)}) &= \sum_{\substack{p \in B_{w_L^+}(\Lambda) \\ p_{L-1} = \Lambda - ((k-i)\widehat{0} + i\widehat{1})}} z^{(\Lambda^{(i)} - (p_0 - (\Lambda - \Lambda^{(i)})) | \Lambda_1)} \\ &\quad \times q^{\sum_{j=1}^{L-1} j(H(\iota(p)_{j-1}, \iota(p)_j) - H(\iota(\bar{p}^{(i)})_{j-1}, \iota(\bar{p}^{(i)})_j))}. \end{aligned}$$

Therefore,

$$\begin{aligned} ch_L^+(\Lambda) &= \sum_{i=0}^s q^{\sum_{j=1}^{L-1} j(H(\iota(\bar{p}^{(i)})_{j-1}, \iota(\bar{p}^{(i)})_j) - H(\iota(\bar{p})_{j-1}, \iota(\bar{p})_j))} \\ &\quad \times q^{L(H(\iota_{L-1}^{(i)}, \iota_L^{(i)}) - H(\iota_{L-1}, \iota_L))} ch_{L-1}(\Lambda^{(i)}). \end{aligned}$$

Here  $\iota_{L-1}^{(i)} = (k-i)\epsilon_0 + i\epsilon_1$ ,  $\iota_{L-1} = \iota_{L-1}^{(s)}$ ,  $\iota_L^{(i)} = \iota_L = s\epsilon_0 + t\epsilon_1$ . Using Lemma 1 we get the result. The case of  $L$  odd is similar.

For  $ch_L^-(\Lambda)$  we utilize the Dynkin diagram symmetry.

$$\begin{aligned} ch_L^-(\Lambda) &= ch_L^+(t\Lambda_0 + s\Lambda_1)(qz^{-1}, q) \\ &= \sum_{i=0}^t (qz^{-1})^{-\epsilon^{(L)}(t-i)} q^{\frac{L+\epsilon^{(L)}}{2}(t-i)} ch_{L-1}(\Lambda^{(i)})(qz^{-1}, q). \end{aligned}$$

Noting that  $ch_{L-1}(\Lambda^{(i)})(qz^{-1}, q) = ch_{L-1}(\Lambda^{(k-i)})(z, q)$ , we get the result. ■

### 3.5 Some specializations

In this subsection, we consider two specializations of Demazure characters, and compare with Sanderson's results [6].

First we consider 'real characters'. These are obtained by setting  $e^{-\alpha_0} = q, e^{-\alpha_1} = q^{-1}$ . We have

**Corollary 3** *Let  $\Lambda = s\Lambda_0 + t\Lambda_1, s + t = k$ .*

$$ch_L^+(\Lambda)(q^{-1}, 1) = q^{-\frac{(L-\epsilon^{(L)})k}{2}} [s+1][k+1]^{L-1}.$$

Here  $[a] = (1 - q^a)/(1 - q)$ .

*Proof.* From Theorem 4 we easily get

$$f_L^{(k)}(b, c; 1) = \sum_{\sum_{i=0}^k ix_i = \frac{Lk-b}{2}} \binom{L}{x_0 \cdots x_k}.$$

Here  $\binom{L}{x_0 \cdots x_k}$  denotes the usual multinomial coefficient. Assume  $L$  is even. From Proposition 7 we have

$$\begin{aligned} ch_L(\Lambda)(q^{-1}, 1) &= \sum_{j \in \mathbb{Z}} q^j \sum_{\sum_{i=0}^k ix_i = \frac{Lk}{2} + j} \binom{L}{x_0 \cdots x_k} \\ &= q^{-\frac{Lk}{2}} \sum_{x_0, \dots, x_k} q^{\sum_{i=0}^k ix_i} \binom{L}{x_0 \cdots x_k} \\ &= q^{-\frac{Lk}{2}} (1 + q + \cdots + q^k)^L \\ &= q^{-\frac{Lk}{2}} [k+1]^L. \end{aligned}$$

Similarly, for  $L$  odd we have

$$ch_L(\Lambda)(q^{-1}, 1) = q^{-\frac{Lk-s+t}{2}}[k+1]^L.$$

Using Theorem 5, we obtain the desired result. ■

Next, we consider ‘principal characters’ obtained by setting  $e^{-\alpha_0} = e^{-\alpha_1} = q$ . In considering this case, we restrict ourselves to the extreme case  $t = 0$ .

**Corollary 4** *Let  $\Lambda = k\Lambda_0$ .*

$$ch_L^+(\Lambda)(q, q^2) = \sum_{x_0, \dots, x_k} q^{2x^t C^{-1}x + \frac{k}{2}\mathcal{S}(\mathcal{S}+1)} \left[ \begin{matrix} L \\ x_0 \cdots x_k \end{matrix} \right]_{q^2},$$

where we have set

$$\mathcal{S} = \frac{1}{k} \sum_{i=0}^k (k-2i)x_i. \quad (18)$$

*Proof.* Note from Corollary 2 that when  $\Lambda = k\Lambda_0$  we have  $ch_L^+(\Lambda) = ch_L(\Lambda)$ . Using Theorem 4 we obtain

$$ch_L^+(\Lambda)(q, q^2) = \sum_{j \in \mathbb{Z}} q^{-j} \sum_{\sum_{i=1}^{k-1} ix_i \in j+k\mathbb{Z}} q^{2x^t C^{-1}x + \frac{2j^2}{k}} \left[ \begin{matrix} L \\ x_0 \cdots x_k \end{matrix} \right]_{q^2}, \quad (19)$$

where

$$x_0 = \left\lfloor \frac{L+1}{2} \right\rfloor - \frac{1}{k} \left( \sum_{i=1}^{k-1} (k-i)x_i + j \right), \quad x_k = \left\lfloor \frac{L}{2} \right\rfloor - \frac{1}{k} \left( \sum_{i=1}^{k-1} ix_i - j \right) \quad (20)$$

with  $\lfloor \cdot \rfloor$  being the Gauss symbol:  $\lfloor x \rfloor$  is the largest integer that does not exceed  $x$ . Since the condition  $\sum_{i=1}^{k-1} ix_i \in j+k\mathbb{Z}$  is equivalent to the condition that all  $x_k$  are integers, we can rewrite the expression in (19) as

$$\sum_{x_0, \dots, x_k} q^{2x^t C^{-1}x + \frac{2j^2}{k} - j} \left[ \begin{matrix} L \\ x_0 \cdots x_k \end{matrix} \right]_{q^2}.$$

Solving (20) with respect to  $j$ , we get

$$j = \frac{k}{2} \left( \left\lfloor \frac{L+1}{2} \right\rfloor - \left\lfloor \frac{L}{2} \right\rfloor - \mathcal{S} \right).$$

Noting that  $\left\lfloor \frac{L+1}{2} \right\rfloor - \left\lfloor \frac{L}{2} \right\rfloor = \epsilon^{(L)}$  and substituting this into the expression, we arrive at

$$\sum_{x_0, \dots, x_k} q^{2x^t C^{-1}x + \frac{k}{2}\mathcal{S}(\mathcal{S}+1-2\epsilon^{(L)})} \left[ \begin{matrix} L \\ x_0 \cdots x_k \end{matrix} \right]_{q^2}.$$

If  $L$  is odd, we can make the variable change  $x_i \rightarrow x_{k-i}$ , which amounts to  $\mathcal{S} \rightarrow -\mathcal{S}$ . Thus we get the desired result. ■

In [6], Sanderson has evaluated the same character involving the  $q$ -multinomial coefficient with argument  $q$ , as opposed to our expression with  $q^2$ . Accordingly, we get the following polynomial identity.

$$\begin{aligned} \sum_{x_0, \dots, x_k} q^{2x^t C^{-1}x + \frac{k}{2}\mathcal{S}(\mathcal{S}+1)} \begin{bmatrix} L \\ x_0 \cdots x_k \end{bmatrix}_{q^2} \\ = \sum_{0 \leq i_1 \leq \dots \leq i_k \leq L} q^{\frac{i_1(i_1+1)}{2} + \dots + \frac{i_k(i_k+1)}{2}} \begin{bmatrix} L \\ L - i_k \ i_k - i_{k-1} \cdots i_2 - i_1 \ i_1 \end{bmatrix}_q, \end{aligned}$$

where  $\mathcal{S}$  is defined in (18).

## 4 Discussion

We characterized all Demazure crystals associated with  $\widehat{sl}(2)$  in terms of paths. We then obtained explicit expressions for their full characters. Our results extend Sanderson's results, and reduce to hers when specialized to certain limits. Our derivations are based on ideas and techniques that originated in computations of certain physical quantities in exactly solvable lattice models, particularly Baxter's corner transfer matrix method, and the combinatorial structures related to it.

Since the very same methods and techniques are available, we expect that our approach can be used for the cases of other affine algebras too. Let us consider the  $\widehat{sl}(n)$  case. In the case we treated in this work, the affine Weyl group  $W$  had two subsets  $W^+$  and  $W^-$  satisfying  $W^+ \cup W^- = W$ ,  $W^+ \cap W^- = \{1\}$ . On each subset the Bruhat order was a total order. But if  $n > 2$  the Weyl group does not have such a property. Thus it is very difficult to treat the Demazure modules corresponding to all Weyl group elements. But for some particular type of elements there does exist a correspondence between Demazure crystals and paths, which we would like to report in near future.

As for 1-dimensional configuration sums in the higher rank case, few results are available except the level 1 case [16]. We would like to push forward these studies. Along this line we note the works by Kuniba-Nakanishi-Suzuki [22] and Georgiev [23]. In [22] they presented conjectures of fermionic expressions for string functions of vacuum modules  $V(k\Lambda_0)$  when the affine algebra is of type  $X_r^{(1)}$ . Georgiev proved the conjecture in the  $A_r^{(1)}$  case. For our purpose we have to 'finitize' their conjecture or result, since we need generating functions of paths of finite length. This would be a challenging problem.

Our approach can be also extended to types other than  $\widehat{sl}(n)$ . In these cases we do not have descriptions of crystals by extended Young diagrams or similar notions yet. We might have to work directly on paths. We have so far

discussed the cases of affine algebra modules. There are definitely the cases of coset modules left. We also would like to understand representation theoretical meaning of the truncated generating functions.

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